Equilibrium Analysis in Cake Cutting

Simina Brânzei and Peter Bro Miltersen
Department of Computer Science
Aarhus University, Denmark
{simina,bromille}@cs.au.dk

ABSTRACT

Cake cutting is a fundamental model in fair division; it represents the problem of fairly allocating a heterogeneous divisible good among agents with different preferences. The central criteria of fairness are proportionality and envy-freeness, and many of the existing protocols are designed to guarantee proportional or envy-free allocations, when the participating agents follow the protocol. However, typically, all agents following the protocol is not guaranteed to result in a Nash equilibrium.

In this paper, we initiate the study of equilibria of classical cake cutting protocols. We consider one of the simplest and most elegant continuous algorithms – the Dubins-Spanier procedure, which guarantees a proportional allocation of the cake – and study its equilibria when the agents use simple threshold strategies. We show that given a cake cutting instance with strictly positive value density functions, every envy-free allocation of the cake can be mapped to a pure Nash equilibrium of the corresponding moving knife game. Moreover, every pure Nash equilibrium of the moving knife game induces an envy-free allocation of the cake. In addition, the moving knife game has an ϵ-equilibrium which is ε-envy-free, allocates the entire cake, and is independent of the tie-breaking rule.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Social and Behavioral Sciences**]: Economics

Keywords

Cake Cutting, Fair Division, Game Theory

1. INTRODUCTION

Cake cutting is a fundamental model in fair division and it represents the problem of allocating a heterogeneous divisible good, such as land, time, or computer memory, in a way so that everyone believes they received a fair amount. The problem has been formally introduced by Steinhaus during the Second World War [15] and has been studied extensively since then in a large body of literature in mathematics, economics and political science (including two books by Brams and Taylor [2], and Robertson and Webb [14]). The central criteria used to determine the fairness of an allocation are proportionality and envy-freeness. Proportionality requires that each participant believes they received their fair share of the cake, while envy-freeness stipulates that no participant prefers someone else's piece to their own. Envy-freeness is a strong requirement which implies proportionality when the entire cake is allocated.

There has been significant research interest in cake cutting in the computer science community, as problems of resource allocation (and fair division, in particular) are central to the design of multiagent systems. Important directions include the study of the complexity of computing envy-free allocations [7], mechanism design [5], richer models inspired by new applications [4] (such as advertising), and the computation of optimal allocations under simple valuation functions [6]. In particular, a recent body of literature has investigated the design of strategy proof mechanisms. The earliest such work is by Brams [1], who considers a weak version of strategy proofness. There, the agents are risk averse and report their true valuations if there exists a choice of valuations of the other agents such that the outcome would be worse by misreporting. Chen *et al.* [5] design strategy-proof mechanisms to compute envy-free and proportional allocations for restricted classes of valuation functions. The main results include a polynomial-time deterministic mechanism which computes an envy-free and proportional allocation for piecewise uniform valuations, and a polynomial-time randomized mechanism which is truthful in expectation, universally proportional, and universally envy-free for piecewise linear valuations. Mossel and Tamuz [13] design an incentive-compatible, proportional, and Pareto-efficient mechanism for general valuations. Maya and Nisan [12] study incentive-compatibility and Pareto-efficiency for two agents, and provide characterizations for mechanisms with such properties.

Going back to the classical cake cutting protocols, it is a standard assumption that the agents do not know each other's preferences. However, in many real-world settings, the participants *do know* each other's preferences. For example, when countries divide land at the end of a war, it is usually common knowledge which areas of land are preferred by which country. Thus, when a general protocol is employed to produce an allocation of the cake, the agents may be able to improve their utility by being strategic if they know the others' valuations during the execution of the al-

Appears in: *Proceedings of the 12th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2013)*, Ito, Jonker, Gini, and Shehory (eds.), May, 6–10, 2013, Saint Paul, Minnesota, USA.

gorithm. In this paper, we initiate the study of equilibria of classical cake cutting protocols.

While the classical protocols are not necessarily strategy-proof, they are often very simple, elegant, and designed so that the agents can easily implement them by following a sequence of natural steps. One of the most intuitive and best-known procedures computes a proportional allocation of the cake and was introduced by Dubins and Spanier [8]. It requires a moving knife and proceeds as follows.

A referee holds a knife and moves it slowly across the cake, from the left to the right endpoint. When the knife reaches a point such that one of the agents has valuation exactly $1/n$ for the piece to the left of the knife, that agent shouts Cut!. *The first agent to do so receives the left piece and exits. The remaining $n-1$ agents repeat the procedure on the leftover cake, except that now they call cut when the perceived value of the piece to the left of the knife is $1/(n-1)$ (of the remaining cake).*

To make the protocol completely precise, a tie breaking rule has to be specified for the case of two players calling cut simultaneously. No matter how such tie breaking is defined, it is easily verified that the allocation produced by the Dubins-Spanier moving-knife procedure is proportional. However, it is not necessarily envy-free.

In this paper, we consider the strategic version of Dubins-Spanier, which we refer to as the *moving knife game*. The agents know each other's valuations and compete against each other to maximize their allocations. In the moving knife game, an agent would like to delay as much as possible the moment of calling cut, since the longer they wait, the better the piece to the left of the knife becomes. However, if they wait for too long, someone else may call cut before they do and take the piece instead. The moving knife game is related to games of timing [11], such as war of attrition models, in which the decision of each agent is when to quit and victory belongs to the agent that held on longer, and preemption games, in which the agents prefer to stop first.

It seems very challenging to characterize all the equilibria of the moving knife game if it is modeled as a continuous time extensive form game in the obvious way. Instead, we analyze the game when the agents are restricted to use *threshold strategies*, defined as follows. The moving knife game proceeds in n rounds, corresponding to each of the time intervals between consecutive cut points. Each agent has n thresholds, one for every round. An agent calls cut in a given round when the value of the piece to the left of the knife is equal to the agent's threshold for that round. Note that threshold strategies is a simple generalization of the prescribed behavior in the orginal Dubins-Spanier protocol – in particular, the classical Dubins-Spanier procedure outlined above can be viewed as playing the moving knife game with all agents using the sequence of thresholds $\left(\frac{1}{n}, \frac{1}{n-1}, \frac{1}{n-2}, \ldots, 1\right)$.

Our main result is a direct correspondence between the equilibria of the moving knife game and envy-free allocations of the cake with contiguous pieces, when players are restricted to threshold strategies.

That is, every pure Nash equilibrium of the moving knife game induces an envy-free allocation with contiguous pieces which contains the entire cake. Moreover, every envy-free allocation with contiguous pieces of the entire cake can be mapped to a pure Nash equilibrium of the corresponding moving knife game, when ties are broken in a particular way. This result can be viewed as an affirmative answer to the natural question: "Can fair allocations arise as equilibria of simple and natural protocols?" The question of designing a game such that its equilibria correspond to desirable allocations of the cake was also considered by Ianovski [10].

2. MODEL

We make more precise the details of the cake cutting model ([2, 14]). Let $N = \{1, \ldots, n\}$ be a set of agents. The cake is modelled as the interval $[0, 1]$. Each agent i has a measurable value density function $v_i : [0, 1] \to \mathbb{R}^+$ which represents their preference for the cake. A *piece* of cake X is a finite union of disjoint subintervals of $[0, 1]$. A *contiguous piece* is a single subinterval. The valuation of agent i for a piece X is given by:

$$V_i(X) = \sum_{I \in X} \int_I v_i(x)dx$$

We assume the value density functions of the agents are normalized, so that each agent has a utility of one when they receive the entire cake:

$$V_i([0, 1]) = \int_0^1 v_i(x)dx = 1$$

An allocation $X = (X_1, \ldots, X_n)$ is an assignment of pieces to agents such that each agent i receives piece X_i and all the X_i are disjoint. The central criteria for determining the fairness of an allocation are proportionality and envy-freeness. An allocation X is *proportional* if

$$V_i(X_i) \geq 1/n, \ \forall i \in N$$

and *envy-free* if

$$V_i(X_i) \geq V_i(X_j), \forall i, j \in N.$$

Note that envy-freeness is a very strong notion, which implies proportionality when the entire cake is allocated.

We now introduce the moving knife game. Given a cake with the corresponding value density functions, a knife moves continuously from the left to the right endpoint of the cake. The game is divided in n rounds. Each agent i has a strategy consisting of n thresholds, $T_i = [t_{i,1}, \ldots, t_{i,n}] \in [0, 1]^n$, one threshold for each round. Agent i calls cut in round j when the piece to the left of the knife is worth exactly $t_{i,j}$ according to i's valuation. The agent to call cut first receives the piece to the left of the knife. When multiple agents call cut simultaneously, the piece is given to the agent who comes first in a tie-breaking rule $\pi = (\pi_1, \ldots, \pi_n)$, which is a fixed permutation of N. Once an agent has received a piece, he exits and the game continues from that point on with the remaining agents and leftover cake.

Given a tuple (N, v, T, π), where N is a set of agents, v are the value density functions, T are the strategies, and π is a tie-breaking rule, the *induced allocation* $X = (X_1, \ldots, X_n)$ results from playing the moving knife game under the tie-breaking rule π, such that each agent $\sigma_i \in N$ receives the piece X_i, for some ordering σ of the agents.

Finally, we say that an agent is *active* at a round if the agent has not exited the game in the previous rounds.

We illustrate the game with the following example.

EXAMPLE 1. *Let $N = \{1, 2\}$. Consider the following value density functions:*

- $v_1(x) = 1$, $\forall x \in [0, 1]$
- $v_2(x) = \frac{1}{4}, \forall x \in \left[0, \frac{1}{3}\right]$ *and* $v_2(x) = \frac{11}{8}, \forall x \in \left[\frac{1}{3}, 1\right]$

Let $T = [T_1, T_2]$, where $T_1 = \left[\frac{1}{2}, \frac{2}{3}\right]$ and $T_2 = \left[\frac{1}{12}, \frac{2}{3}\right]$. Then in:

- Round 1*: Agent 2 calls cut first at $\frac{1}{3}$, since $V_2\left(\left[0, \frac{1}{3}\right]\right) = t_{2,1} = \frac{1}{12}$. Agent 1 does not get to call cut in this round, since $V_1\left(\left[0, \frac{1}{3}\right]\right) < t_{1,1} = \frac{1}{2}$.*
- Round 2*: Agent 1 is the only one left, and the leftover cake is $\left[\frac{1}{3}, 1\right]$. Agent 1 calls cut at 1, since $V_1\left(\left[\frac{1}{3}, 1\right]\right) = t_{2,1} = \frac{2}{3}$.*

The induced allocation is $X = (X_1, X_2)$, where agent 2 receives $X_1 = \left[0, \frac{1}{3}\right]$ and agent 1 receives $X_2 = \left[\frac{1}{3}, 1\right]$.

A strategy profile $T = [T_1, \ldots, T_n] \in [0, 1]^{n \times n}$ is a *pure Nash equilibrium* under a tie-breaking rule π if no agent $i \in N$ can receive a better allocation by deviating to $T_i^{'} \neq T_i$. That is, $u_i(T) \geq u_i(T_i^{'}, T_{-i}), \forall T_i^{'} \in [0, 1]^n$.

In the following example, we illustrate how the agents can be strategic during the execution of the moving knife game, i.e., we illustrate that it is not necessarily a Nash equilibrium that all agents play the Dubins-Spanier strategy $[1/n, \ldots, 1/2, 1]$. Consider the scenario where agent 1 has a uniform valuation over the cake and just wants as much of it as possible, while agent 2 only likes a very thin slice at the right end. Then agent 1 can delay the moment of calling cut, since he knows that agent 2 is following the Dubins-Spanier recommendation and will only call cut close to the right endpoint. A precise version of this example follows:

EXAMPLE 2. *Let $N = \{1, 2\}$. Consider the following value density functions:*

- $v_1(x) = 1$, $\forall x \in [0, 1]$
- $v_2(x) = 0$, $\forall x \in \left[0, \frac{3}{4}\right]$ *and* $v_2(x) = 4$, $\forall x \in \left[\frac{3}{4}, 1\right]$.

Under the Dubins-Spanier protocol, agent 1 calls cut first at $\frac{1}{2}$. The resulting allocation is $X = (X_1, X_2)$, with $X_1 = \left[0, \frac{1}{2}\right]$ and $X_2 = \left[\frac{1}{2}, 1\right]$, with utilities: $V_1(X_1) = \frac{1}{2}$ and $V_2(X_2) = 1$.

However, agent 1 can improve his utility by waiting and calling cut at $\frac{3}{4}$ instead. Then the allocation is $X^{'} = (X_1^{'}, X_2^{'})$, with $X_1^{'} = \left[0, \frac{3}{4}\right]$ and $X_2^{'} = \left[\frac{3}{4}, 1\right]$. The new utilities are $V_1(X_1^{'}) = \frac{3}{4}$ and $V_2(X_2^{'}) = 1$.

3. EXACT EQUILIBRIA

In this section, we analyze the pure Nash equilibria of the moving knife game, for any fixed strictly positive value density functions (i.e. $v_i(x) > 0$, $\forall x \in [0, 1]$, $\forall i \in N$).

First, the original result of Dubins and Spanier immediately yields the following proposition.

PROPOSITION 1. *In any pure Nash equilibrium of the moving knife game, each agent's utility is at least $1/n$ and the entire cake is allocated to the agents.*

PROOF. Suppose a player gets a smaller utility in Nash equilibrium. Then he can deviate by playing the strategy prescribed in the original Dubins-Spanier protocol, i.e., $[1/n, \ldots, 1/2, 1]$, improving his utility to at least $1/n$, and contraditing that a Nash equilibrium is played. Also, if the entire cake is not allocated, the last player's last threshold is strictly smaller than 1. He can therefore deviate to threshold 1 and receive a larger utility, contradicting that a Nash equilibrium is played. □

Now we show that the existence of Nash equilibrium crucially depends on the tie breaking rule used. That is, there exist tie-breaking rules where the moving knife game does not have a pure Nash equilibrium:

PROPOSITION 2. *There exist a tie breaking rules and value density functions so that the corresponding moving knife game does not have a pure Nash equilibrium.*

PROOF. Let $N = \{1, 2\}$, with tie-breaking order $(1, 2)$, and value density functions:

- $v_1(x) = \frac{1}{4}$, $\forall x \in \left[0, \frac{4}{5}\right]$ and $v_1(x) = 37.5x - 29.75$, $\forall x \in \left[\frac{4}{5}, 1\right]$
- $v_2(x) = 1$, $\forall x \in [0, 1]$.

Assume there exists a profile of threshold strategies in equilibrium, T, such that the first cut is made at $x \in (0, 1]$. We analyze the case where the cut at x is made in round 1; the case where the cut is made in round 2 is similar.

First, T must be such that both agents call cut at x simultaneously. Otherwise, if $t_{1,1} = V_1([0, x])$, while $t_{2,1} < V_2([0, x])$, then agent 1 can increase his threshold to $t_{1,1}^{'} = t_{1,1} + \varepsilon$, for small enough $\varepsilon > 0$, and receive a strictly better piece, $[0, x^{'}]$, where $x^{'} > x$.

Similarly, if $t_{2,1} = V_2([0, x])$, while $t_{1,1} > V_1([0, x])$, then agent 2 can increase his threshold and get a strictly better piece $[0, x^{'}]$. Thus $t_{1,1} = V_1([0, x])$ and $t_{2,1} = V_2([0, x])$. Since the tie-breaking rule is $(1, 2)$, agents 1 and 2 receive pieces $[0, x]$ and $[x, 1]$, respectively.

In addition, we have that:

$$V_1([0, x]) \geq V_1([x, 1]), \tag{1}$$

since otherwise agent 1 can deviate by setting $t_{1,1} = 1$ – the deviation would ensure that agent 1 receives a better piece in round 2. Similarly, it can be shown that:

$$V_2([x, 1]) \geq V_2([0, x]). \tag{2}$$

However, inequalities (1) and (2) cannot be met simultaneously for the given valuations. Thus the pure Nash equilibrium T cannot exist. □

We show that in a pure Nash equilibrium, then in each of the first $n - 1$ rounds, the agent who is allocated a piece has a competitor that calls cut simultaneously in that round.

PROPOSITION 3. *Let a moving knife game with strictly positive value density functions be given. Let T be a profile of threshold strategies in equilibrium under a deterministic tie-breaking rule. Then, in every round except the last one, the agent who is allocated the piece has an (active) competitor that calls cut simultaneously.*

PROOF. Let $X = (X_1, \ldots, X_n)$ be the allocation induced by T, such that agent σ_i receives the piece $X_i = [x_{i-1}, x_i]$. It follows by Proposition 1 that X contains the entire cake and $X_i \neq \emptyset$, $\forall i \in N$. Assume by contradiction that there exists a round $i < n$ in which only agent σ_i calls cut at x_i. Then it must be the case that

$$t_{\sigma_j,i} > V_{\sigma_j}([x_{i-1}, x_i]), \forall \sigma_j \in N \setminus \{\sigma_i\}.$$

By the continuity of the valuation functions, there exists $\varepsilon > 0$ such that by deviating to threshold:

$$t'_{\sigma_i,i} = t_{\sigma_i,i} + \varepsilon$$

in round i, agent σ_i is guaranteed a strictly better piece, $[x_{i-1}, x'_i]$, where $x'_i > x_i$. This is a contradiction with T being in equilibrium. Thus every agent who receives a piece in the first $n-1$ rounds has a competitor that calls cut simultaneously in that round. □

Finally, every profile of threshold strategies in equilibrium induces an envy-free allocation. We first show the following proposition.

PROPOSITION 4. *Let a moving knife game with strictly positive value density functions be given. Let T be a profile of threshold strategies under a deterministic tie-breaking rule, such that each agent σ_i receives a piece in round i and in every round except the last, there exist two active agents who call cut simultaneously. Then if some agent σ_i deviates to $T'_{\sigma_i} \neq T_{\sigma_i}$ and receives a new piece in some round k under $T' = (T'_{\sigma_i}, T_{-\sigma_i})$, then the set of cuts made in the first $k-1$ rounds are the same under T and T'.*

PROOF. Let $X = (X_1, \ldots, X_n)$ be the allocation induced by T, where the piece $X_i = [x_{i-1}, x_i]$ is given to agent σ_i. Let T'_{σ_i} be the new sequence of thresholds used by agent σ_i, where

$$t'_{\sigma_i,k} = 1, \forall k \in \{1, \ldots, j\}$$

Since agent σ_i did not receive a piece in the first $i-1$ rounds under T, and does not call cut before other agents under $T' = (T'_{\sigma_i}, T_{-i})$, it follows that the allocation X' (induced by T') is identical to X for the first $i-1$ pieces. If $j < i$, then the statement of the proposition follows immediately.

Otherwise, $j \geq i$. By condition 3 of the proposition, there exists an agent $\sigma_{r_1} \neq \sigma_i$ who also calls cut at x_i in round i, and is second after σ_i in the tie-breaking rule among the agents that call cut at x_i. That is,

$$t_{\sigma_{r_1},i} = V_{\sigma_{r_1}}([x_{i-1}, x_i]).$$

Then agent σ_{r_1} receives the piece $[x_{i-1}, x_i]$ under T'.

The allocations made in rounds $i+1, \ldots, r_1 - 1$ are identical under T' and T, since the same agents that received the pieces

$$X_{i+1}, \ldots, X_{r_1-1}$$

under T continue to call cut at the points:

$$x_{i+1}, \ldots, x_{r_1-1},$$

respectively, and to win the ties (if any) under T'. The piece X_{r_1} is taken by some agent σ_{r_2}, which was second in the tie for receiving the piece X_{r_1} under T.

Iteratively, it can be shown that in the first j rounds, the same cuts are made under T and T', and this set is $\{x_1, \ldots, x_j\}$. □

THEOREM 1. *Consider a moving knife game with strictly positive value density functions and deterministic tie-breaking. Then every pure Nash equilibrium of the game induces an envy-free allocation.*

PROOF. Let T be a profile of threshold strategies in equilibrium under tie-breaking rule $\pi = (\pi_1, \ldots, \pi_n)$. Let $X = (X_1, \ldots, X_n)$ be the induced allocation, such that piece $X_i = [x_{i-1}, x_i]$ is given to agent σ_i, $\forall i \in N$.

Assume by contradiction that X is not envy-free. Since the empty allocation is envy-free, it follows by Proposition 1 that X contains the entire cake. Then there exists an agent σ_i such that

$$V_{\sigma_i}([x_{j-1}, x_j]) > V_{\sigma_i}([x_{i-1}, x_i]),$$

for some $j \in N \setminus \{i\}$. By continuity of the valuation functions, there exists $\varepsilon > 0$ such that

$$V_{\sigma_i}([x_{j-1}, x_j - \varepsilon]) > V_{\sigma_i}([x_{i-1}, x_i]).$$

We consider two cases:

1. $(j < i)$: Then agent σ_i can deviate to strategy profile T'_{σ_i}, where

$$t'_{\sigma_i,k} = \begin{cases} V_{\sigma_i}([x_{j-1}, x_j - \varepsilon]) & \text{if k = j} \\ t_{\sigma_i,k} & \text{otherwise} \end{cases}$$

   Under $T' = (T'_{\sigma_i}, T_{-\sigma_i})$, agent σ_i is guaranteed to receive the piece $[x_{j-1}, x_j - \varepsilon]$, since no other agent calls cut before x_j in round j. This deviation improves σ_i's utility, contradiction with T being in equilibrium.

2. $(j > i)$: Then agent σ_i can deviate to strategy profile T'_{σ_i}, where

$$t'_{\sigma_i,k} = \begin{cases} V_{\sigma_i}([x_{j-1}, x_j - \varepsilon]) & \text{if k = j} \\ 1 & \text{otherwise} \end{cases}$$

   By Proposition 4, the same cuts are made under T and T' in the first $j-1$ rounds, and this set is $\{x_1, \ldots, x_{j-1}\}$.

   Then agent σ_i receives the piece $[x_{j-1}, x_j - \varepsilon]$ in round j, which strictly improves σ_i's utility, since:

$$V_{\sigma_i}([x_{j-1}, x_j - \varepsilon]) > V_{\sigma_i}([x_{i-1}, x_i]).$$

   This is a contradiction with T being in equilibrium.

From Case 1 and 2, it follows that the assumption must have been false, and so the induced allocation is envy-free. □

We can now characterize the set of pure Nash equilibria as follows.

THEOREM 2. *Consider a moving knife game with strictly positive value density functions. A strategy profile T is in Nash equilibrium under a deterministic tie-breaking rule if and only if the induced allocation contains the entire cake and is envy-free and in every round except the last one, the agent who is allocated the piece has an active competitor that calls cut simultaneously.*

PROOF. Let T be a profile of thresholds strategies.

($\Rightarrow$): If T is a pure Nash equilibrium under some deterministic tie-breaking rule, then by Proposition 1 and Theorem 1, it follows that the induced allocation contains the entire cake and is envy-free. Also, by Proposition 3, in every round except the last, the agent who is allocated the piece has an active competitor that calls cut simultaneously.

($\Leftarrow$): If T verifies the conditions of the theorem, then we claim it is a pure Nash equilibrium. Let $X = (X_1, \ldots, X_n)$ be the induced allocation, where piece $X_i = [x_{i-1}, x_i]$ is given to agent σ_i, $\forall i \in N$.

Assume by contradiction that there exists an agent σ_i who can improve by deviating to $T^{'}_{\sigma_i} \neq T_{\sigma_i}$. Let k be the round in which σ_i receives a piece when playing $T^{'}_{\sigma_i}$. Since σ_i does not receive a piece in the first $k-1$ rounds under $T^{'}_{\sigma_i}$, we can assume without loss of generality that:

$$t^{'}_{\sigma_i,l} = 1, \forall l \in \{1, \ldots, k-1\}.$$

By Proposition 4, the cut made in round $k-1$ was at x_{k-1}, and one of the following conditions holds:

- $x_k = 1$, or
- there exists an agent $\sigma_j \neq \sigma_i$ who calls cut at x_k (in round k) when σ_i plays $T^{'}_{\sigma_i}$.

Thus the highest value that σ_i can receive in round k is $V_{\sigma_i}(X_k)$. By envy-freeness of X, we have that:

$$V_{\sigma_i}(X_i) \geq V_{\sigma_i}(X_k),$$

thus the deviation does not improve σ_i's utility. Thus, T is an equilibrium, which concludes the proof of the theorem. □

Next, we show that for every moving knife game with strictly positive value density functions, a pure Nash equilibrium is guaranteed to exist for *some* deterministic tie-breaking rule. In fact, we show that for any envy-free allocation of the cake, there exists is a pure Nash equilibrium that induces this allocation. This implies existence of a pure Nash equilbrium, as an envy-free allocation of the cake with $n-1$ cuts is guaranteed to exist (see Stromquist [16]).

THEOREM 3. *Considser a moving knife game with strictly positive value density functions. Given any envy free allocation of the cake with $n-1$ cuts, there exists a deterministic tie-breaking rule π such that the game has a pure Nash equilibrium inducing this allocation.*

PROOF. In an envy free allocation of the cake with $n-1$ cuts, each agent gets a contigous pice. That is, there exists a permutation $\pi = (\pi_1, \ldots, \pi_n)$ of N and numbers x_i such that agent π_i receives the piece $X_i = [x_{i-1}, x_i]$. Now use π as the tie-breaking rule for the moving knife game and consider the strategy sets:

$$T_i = [t_{i,1}, \ldots, t_{i,n}],$$

where

$$t_{i,k} = V_i([x_{k-1}, x_k]), \forall i, j \in N.$$

It can be verified that the strategies in T verify the conditions of Theorem 2. That is, the induced allocation is envy-free, contains the entire cake, and in every round except the last, the agent winning the piece has a competitor who calls cut simultaneously. Thus the set of strategies T are in equilibrium under π. □

This completes the proof of our main result: Any pure Nash equilibrium of the moving nice game induces an envy-free allocation and any envy-free allocation is induced by some pure Nash equilibrium.

4. ACHIEVING TIE BREAKING RULE INDEPENDENCE

The dependence of the existence of Nash equilibrium on the tie breaking rule is an annoying (but unavoidable) flaw of our main result: The tie-breaking rule requires information about the valuation functions of the agents in order for a non-trivial pure Nash equilibrium to exist. Clearly, in many natural settings, the tie-breaking rule is given exogenously. For example, when countries divide land at the end of a war, some countries may have higher priority than others due to prior bilateral agreements that have been signed.

It is interesting to understand the special cases where a pure Nash equilibrium is guaranteed to exist, no matter which tie breaking rule is used. We have first the following simple observation.

PROPOSITION 5. *Consider a moving knife game with agents with* identical *positive value density functions. Then the game has a pure Nash equilibrium under* every *deterministic tie-breaking rule.*

PROOF. Consider an envy-free division of the cake with $n-1$ cuts, $[x_0, \ldots, x_n]$. The agents have identical value density functions, and so

$$V_i([x_{j-1}, x_j]) = \frac{1}{n}, \forall i, j \in N.$$

For any tie-breaking rule π, construct an allocation $X = (X_1, \ldots, X_n)$, such that agent π_i receives the piece $X_i = [x_{i-1}, x_i]$. By applying Theorem 3 to the envy-free allocation X, it follows that the game has a pure Nash equilibrium under π. □

Next, we show that for arbitrary strictly positive value density functions and every possible tie-breaking rule, including, for example, randomized or round-dependent rules, there exists an *approximate* equilibrium in pure strategies such that the induced allocation is *approximately* envy-free and contains the entire cake.

We say that a set of strategies $T = [T_1, \ldots, T_n] \in [0,1]^{n \times n}$ is an ε-equilibrium if for every $i \in N$, agent i cannot improve his utility by more than ε by deviating to $T^{'}_i \neq T_i$. That is, $u_i(T^{'}_i, T_{-i}) \leq u_i(T) + \varepsilon$.

THEOREM 4. *Consider a moving knife game with strictly positive value density functions. Then for every tie-breaking rule, the game has an ε-equilibrium in pure strategies such that the induced allocation is ε-envy-free and contains the entire cake.*

PROOF. Let $\varepsilon > 0$ and $X = (X_1, \ldots, X_n)$ an envy-free allocation of the entire cake, where agent π_i receives the piece $X_i = [x_{i-1}, x_i]$, $\forall i \in N$.

Starting from X, we construct an allocation $Z = (Z_1, \ldots, Z_n)$, where agent π_i receives the piece $Z_i = [z_{i-1}, z_i]$, $\forall i \in N$, such that Z is induced by an ε-equilibrium T, contains the entire cake, and is ε-envy-free. The idea of the proof is similar to that of Theorem 3. To avoid tie-breaking, we construct the thresholds such that for every round, the active agents would call cut immediately *after* the agent who is supposed to receive an allocation in that round. Thus, if an agent π_i deviates to a new sequence of thresholds T'_{π_i} and receives a new piece in round $k \neq i$, then the set of cuts made in rounds $\{1, \ldots, k-1\}$ are *approximately the same* under T and $T' = (T'_{\pi_i}, T_{-\pi_i})$. That is, the following hold for the allocation induced by T':

- If π_i still receives a piece in round i, then the improvement cannot be larger than ε, since another active agent will call cut immediately after π_i's expected cut point in T.

- If π_i receives a piece in round $k < i$, then π_i's new piece is a subset of Z_k, and so the improvement cannot be greater than ε by ε-envy-freeness of Z.

- If π_i receives a piece in round $k > i$, then the set of cuts made in rounds $\{1, \ldots, k-1\}$ are approximately the same under T and T', and so π_i's new piece is approximately a subset of Z_k. Again the improvement cannot be better than ε by ε-envy-freeness of Z.

Formally, the profile of threshold strategies T is defined as follows. Let $z_n = x_n$. The valuation functions are continuous and bounded, thus there exists $z_{n-1} \in (x_{n-2}, x_{n-1})$ such that:

$$V_j([z_{n-1}, x_{n-1}]) < \frac{\varepsilon}{2}, \forall j \in N.$$

We construct a set of points:

$$y_{1,k}, \ldots, y_{n,k}, z_k$$

for all rounds k, such that the threshold of each agent j is set to call cut at $y_{j,k} \in [z_{k-1}, x_{k-1})$ in round k. Define $y_{j,n} = z_n$, $\forall j \in N$.

Consider round $n-1$, and let

$$y_{j,n-1} = \begin{cases} z_{n-1} & \text{if } j = \pi_{n-1} \\ \frac{z_{n-1}+x_{n-1}}{2} & \text{otherwise} \end{cases}$$

We now construct z_{n-2}. For each $j \in N$, there exists

$$z_{j,n-2} \in (x_{n-3}, x_{n-2})$$

such that

$$V_j\left([z_{j,n-2}, x_{n-2}]\right) < V_j([y_{j,n-1}, x_{n-1}])$$

Define $z_{n-2} = \max_{j \in N} z_{j,n-2}$. For all $j \in N$, we have:

$$\begin{aligned} V_j([z_{n-2}, x_{n-2}]) &< V_j([y_{j,n-1}, x_{n-1}]) \\ &< V_j([z_{n-1}, x_{n-1}]) < \frac{\varepsilon}{2} \end{aligned}$$

Iteratively, for all rounds k from $n-2$ to 1, we construct points

$$y_{1,k}, \ldots, y_{n,k}, z_{k-1}$$

in a manner similar to the construction for round $n-1$, such that the following conditions are met:

- $z_{k-1} < x_{k-1}$, if $k \in \{2, \ldots, n-1\}$, and $z_{k-1} = x_{k-1}$, if $k = 1$

- $x_{k-1} < z_k \leq y_{1,k}, \ldots, y_{n,k} < x_k$

- $V_j([z_{k-1}, x_{k-1}]) < V_j([y_{j,k}, x_k])$, $\forall j \in N$.

Consider the profile of threshold strategies T, given by:

$$t_{j,k} = V_j([z_{k-1}, y_{j,k}]), \forall j, k \in N$$

Let Z be the allocation induced by T, where agent π_i receives the piece $Z_i = [z_{i-1}, z_i]$, $\forall i \in N$. We claim that T is an ε-equilibrium and Z is ε-envy-free.

First, we show that T is an ε-equilibrium. Assume by contradiction that there exists agent π_i who can improve his utility more than ε by deviating to T'_{π_i}. Let k be the round in which agent π_i is allocated a piece under $T' = (T'_{\pi_i}, T_{-\pi_i})$. We show by induction that in each previous round $l < k$, a cut is made in the interval $[z_l, x_l)$. For $l = 1$ the statement trivially holds, since:

$$0 < t_{j,1} = V_j([0, y_{j,1}]) < V_j([0, x_1]), \forall j \in N$$

Assume the property holds for all rounds $1, \ldots, l-1$. By the induction hypothesis, a cut was made in round $l-1$ in the interval $[z_{l-1}, x_{l-1})$. For each agent j, the threshold in round l is such that:

$$\begin{aligned} t_{j,l} &= V_j([z_{l-1}, y_{j,l}]) \\ &= V_j([z_{l-1}, x_{l-1}]) + V_j([x_{l-1}, x_l]) - V_j([y_{j,l}, x_l]) \\ &< V_j([x_{l-1}, x_l]) \end{aligned}$$

Note that the inequality:

$$V_j([z_{l-1}, x_{l-1}]) < V_j([y_{j,l}, x_l])$$

holds by Condition 3. Thus, in round l, every remaining agent j will call cut

- no earlier than $y_{j,l}$ if in the previous round the cut was made at z_{l-1}

- strictly before x_l if in the previous round the cut was made at x_{l-1}

Thus the statement also holds for round l. It follows that the cut in round $k-1$ was made in the interval $[z_{k-1}, x_{k-1})$. Moreover, all the remaining agents will call cut before x_k in round k. Then, using the envy-freeness of allocation X, we can bound the utility of π_i as follows:

$$\begin{aligned} u_i(T') &< V_{\pi_i}([z_{k-1}, x_k]) \\ &= V_{\pi_i}([z_{k-1}, x_{k-1}]) + V_{\pi_i}([x_{k-1}, x_k]) \\ &\leq \frac{\varepsilon}{2} + V_{\pi_i}([x_{i-1}, x_i]) \\ &\leq \frac{\varepsilon}{2} + V_{\pi_i}([x_{i-1}, x_i]) + \\ &+ \left(V_{\pi_i}([z_{i-1}, x_{i-1}]) - V_{\pi_i}([z_i, x_i]) + \frac{\varepsilon}{2}\right) \\ &= V_{\pi_i}([z_{i-1}, z_i]) + \varepsilon \\ &= u_i(T) + \varepsilon \end{aligned}$$

Thus agent i cannot improve by more than ε by deviating.

Finally, we show that the induced allocation is ε-envy-free. For every two agents π_i and π_j the following hold:

$$\begin{aligned} & V_{\pi_i}(Z_i) = V_{\pi_i}([z_{i-1}, z_i]) \\ = \; & V_{\pi_i}([x_{i-1}, x_i]) + V_{\pi_i}([z_{i-1}, x_{i-1}]) - V_{\pi_i}([z_i, x_i]) \\ \geq \; & V_{\pi_i}([x_{j-1}, x_j]) - \frac{\varepsilon}{2} \\ \geq \; & V_{\pi_i}([z_{j-1}, z_j]) + V_{\pi_i}([z_j, x_j]) - V_{\pi_i}([z_{j-1}, x_{j-1}]) - \frac{\varepsilon}{2} \\ \geq \; & V_{\pi_i}([z_{j-1}, z_j]) - \varepsilon \\ = \; & V_{\pi_i}(Z_j) - \varepsilon \end{aligned}$$

where the inequality:

$$V_{\pi_i}([x_{i-1}, x_i]) \geq V_{\pi_i}([x_{j-1}, x_j])$$

holds by envy-freeness of X. Thus the induced allocation, Z, is ε-envy-free and contains the entire cake. □

5. THE GENERALIZED GAME

In this section we introduce and briefly discuss a natural generalization of the moving knife game, in which the agents can receive multiple pieces of cake. This generalization is motivated by several other moving knife procedures [14] in which the agents can receive more than one piece of cake (see, e.g., the moving knife scheme of Brams *et al.* [3], which can use as many as eleven cuts to produce an envy-free allocation for four agents).

Informally, a *generalized moving knife game* is a moving knife game where each agent $i \in N$ can receive up to $m_i \in \mathbb{N}^*$ pieces, and the game has $M \in \mathbb{N}^*$ rounds. In the generalized game, a strategy of agent i consists of a sequence of M thresholds:

$$T_i = [t_{i,1}, \ldots, t_{i,M}] \in [0,1]^M,$$

such that agent i calls cut in round k when the piece to the left of the knife is worth $t_{i,k}$ according to i's valuation. The moving knife game introduction in Section 2 is an instance of the generalized game where the budget of each agent is one and the number of rounds is n. A particularly relevant instance of the generalized moving knife game is the one-round moving knife game (with $M = 1$ and $m_i = 1, \forall i \in N$), which is related to war of attrition models (see, e.g., the war of attrition in continuous time analyzed by Hendricks *et al* [9]).

In the case of one-round moving knife games with strictly positive value density functions, this is the unique pure Nash equilibrium.

Proposition 6. *In a one-round moving knife game with strictly positive value density functions, every pure Nash equilibrium of the game induces the empty allocation.*

Proof. Assume by contradiction that there exists a one-round game with strictly positive value density functions, continuous valuations, and deterministic tie-breaking such that the game has a non-trivial pure Nash equilibrium. Without loss of generality, let us assume that the tie-breaking rule is $(1, \ldots, n)$.

Let T be a profile of threshold strategies in equilibrium. Then there exists $x \in [0,1]$ such that $V_i([0,x]) = t_i$ for some agent $i \in N$, and the following hold:

- $V_j([0,x]) < t_j, \forall j \in \{1, \ldots, i-1\}$
- $V_j([0,x]) \leq t_j, \forall j \in \{i+1, \ldots, n\}$.

The utilities under T are:

$$u_i(T) = t_i$$

and

$$u_j(T) = 0, \forall j \in N \setminus \{i\}.$$

Then any agent $j \in N \setminus \{i\}$, can strictly improve their utility by deviating to threshold:

$$t_j^{'} = \frac{V_j([0, x/2])}{2},$$

since

$$u_j(T^{'}, T_{-j}) = t_j^{'} > 0,$$

where $T^{'} = (T_j^{'}, T_{-j})$. □

More generally, the result holds for all moving knife games with strictly positive value density functions where the number of rounds is small enough (i.e. $M < \sum_{i=1}^{n} m_i$).

Finally, when the agents have symmetric value density functions, i.e. $v_i(x) = v_j(x), \forall i, j \in N$, and the number of rounds is large enough to allow all the agents to receive the number of pieces they are entitled to, then the generalized moving knife game has a non-trivial pure Nash equilibrium for every deterministic tie-breaking rule.

Proposition 7. *Consider a generalized moving knife game with symmetric and strictly positive value density functions, where the number of rounds is equal to the total number of pieces that the agents are entitled to. Then the game has a pure Nash equilibrium for every deterministic tie-breaking rule.*

Proof. Let M be the number of rounds and m_i the maximum number of pieces that agent i is entitled to receive. Then we have that $M = \sum_{i=1}^{n} m_i$.

Let $\pi = (\pi_1, \ldots, \pi_n)$ be the tie-breaking rule. Since the agents have identical value density functions, there exists a partition of the cake in M contiguous pieces, $X = (X_1, \ldots, X_M)$, such that

$$V_i(X_j) = \frac{1}{M}, \forall i \in N.$$

Define the following thresholds:

$$t_{i,k} = \frac{1}{M}, \forall i \in N, k \in \{1, \ldots, M\}.$$

It can be easily verified that the strategies are in equilibrium, and the utility of each agent under T is:

$$u_i(T) = \frac{m_i}{M}, \forall i \in N.$$

Note that the equilibrium allocation of each agent is directly influenced by their budget, i.e. agents with higher budget receive proportionally larger pieces. □

6. DISCUSSION AND FUTURE WORK

We studied the strategic version of the Dubins-Spanier protocol when the agents have simple threshold strategies. Our main technical result is the existence of a direct correspondence between the non-trivial pure Nash equilibria of the moving knife game and the envy-free allocations of the cake with contiguous pieces. A characterization of the equilibria in the generalized moving knife game is left open. If one requires that the induced allocations have desirable properties, related to proportionality and envy-freeness, then the existence of such equilibria depends on whether envy-free allocations with a given number of cuts and ordering of the agents exist. In particular, we are interested in the existence of mixed-strategy equilibria with uncountably infinite support, such that the entire cake is allocated with positive probability.

It would also be interesting to understand the outcomes of the game under richer strategy spaces. We note that generalizations in which each agent has $n!$ thresholds (to account not only for the round number, but also for the players that have been allocated in the previous rounds) do not necessarily have envy-free equilibria. However, this does not preclude the existence of envy-free equilibria in the corresponding continuous time extensive form game.

In addition, this work initiates the direction of understanding the consequences of strategic behaviour in classical cake cutting protocols. For example, it would be interesting to understand whether protocols that compute fair allocations in the classical model (such as Brams-Talor) have fair equilibria under complete information.

7. ACKNOWLEDGEMENTS

We would like to thank Margareta Ackerman, Ariel Procaccia, and the anonymous AAMAS reviewers for valuable feedback that helped improve the paper.

This work was supported by the Sino-Danish Center for the Theory of Interactive Computation, funded by the Danish National Research Foundation and the National Science Foundation of China (under the grant 61061130540). The authors also acknowledge support from the Center for research in the Foundations of Electronic Markets (CFEM), supported by the Danish Strategic Research Council.

8. REFERENCES

[1] S. J. Brams, M. A. Jones, and C. Klamler. Better ways to cut a cake. *Notices of the AMS*, 53(11):1314–1321, 2006.

[2] Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.

[3] Steven J. Brams, Alan D. Taylor, and William S. Zwicker. A moving knife solution to the four person envy-free cake-division problem. In *Proceedings of the American Mathematical Society*, pages 547–554, 1997.

[4] I. Caragiannis, J. K. Lai, and A. D. Procaccia. Towards more expressive cake cutting. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 127–132, 2011.

[5] Yiling Chen, John K. Lai, David C. Parkes, and Ariel D. Procaccia. Truth, Justice, and Cake Cutting. *Games and Economic Behavior*, 77:284–297, 2013.

[6] Yuga J. Cohler, John K. Lai, David C. Parkes, and Ariel D. Procaccia. Optimal Envy-Free Cake Cutting. In *Proceedings of the 25th Conference on Artificial Intelligence (AAAI)*, pages 626–631, 2011.

[7] Xiaotie Deng, Qi Qi, and Amin Saberi. Algorithmic Solutions for Envy-Free Cake Cutting. *Operations Research*, 60(6):1461–1476, 2012.

[8] L. E. Dubins and E. H. Spanier. How to cut a cake fairly. *The American Mathematical Monthly*, 68(1), 1961.

[9] Ken Hendricks, Andrew Weiss, and Charles A. Wilson. The War of Attrition in Continuous Time with Complete Information. *International Economic Review*, 29(4):663–680, 1988.

[10] Egor Ianovski. Cake Cutting Mechanisms. *http://arxiv.org/abs/1203.0100*, 2012.

[11] Rida Laraki, Eilon Solan, and Nicolas Vieille. Continuous-time games of timing. *Journal of Economic Theory*, 129:206–238, 2005.

[12] Avishay Maya and Noam Nisan. Incentive Compatible Two Player Cake Cutting. In *Proceedings of the 8th Workshop on Internet and Network Economics (WINE)*, 2012.

[13] E. Mossel and O. Tamuz. Truthful fair division. In *Proceedings of the 3rd International Symposium on Algorithmic Game Theory (SAGT)*, pages 288–299, 2010.

[14] Jack Robertson and William Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A K Peters/CRC Press, 1998.

[15] Hugo Steinhaus. The problem of fair division. *Econometrica*, (16):101–104, 1948.

[16] Walter Stromquist. How to cut a cake fairly. *American Mathematical Monthly*, (8):640–644, 1980. Addendum, vol. 88, no. 8 (1981). 613-614.