Externalities in Cake Cutting

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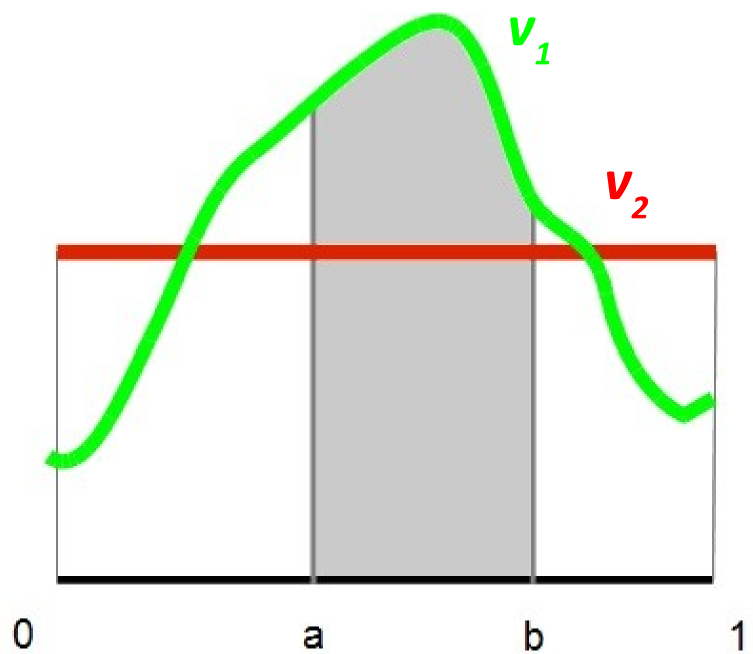
Cake Cutting

- Basic problem in resource allocation
- Models the fair allocation of a divisible heterogeneous resource (land, time, computer memory)
- Studied since the 1940's (Banach, Knaster, Steinhaus) in mathematics, political science, economics

Model

- Set of agents $N = \{1,\ldots, n\}$; the cake is the interval $[0, 1]$
- Each agent i has a valuation function V_i over the cake, which is the integral of a value density function v_i
- An **allocation** $A = (A_1, \ldots, A_n)$ is an assignment of (disjoint) pieces to the agents



Fairness Criteria

Proportionality

- Allocation *A* is proportional if each agent gets a fair share of the cake: $V_i(A_i) \geq 1/n, \forall i \in N$

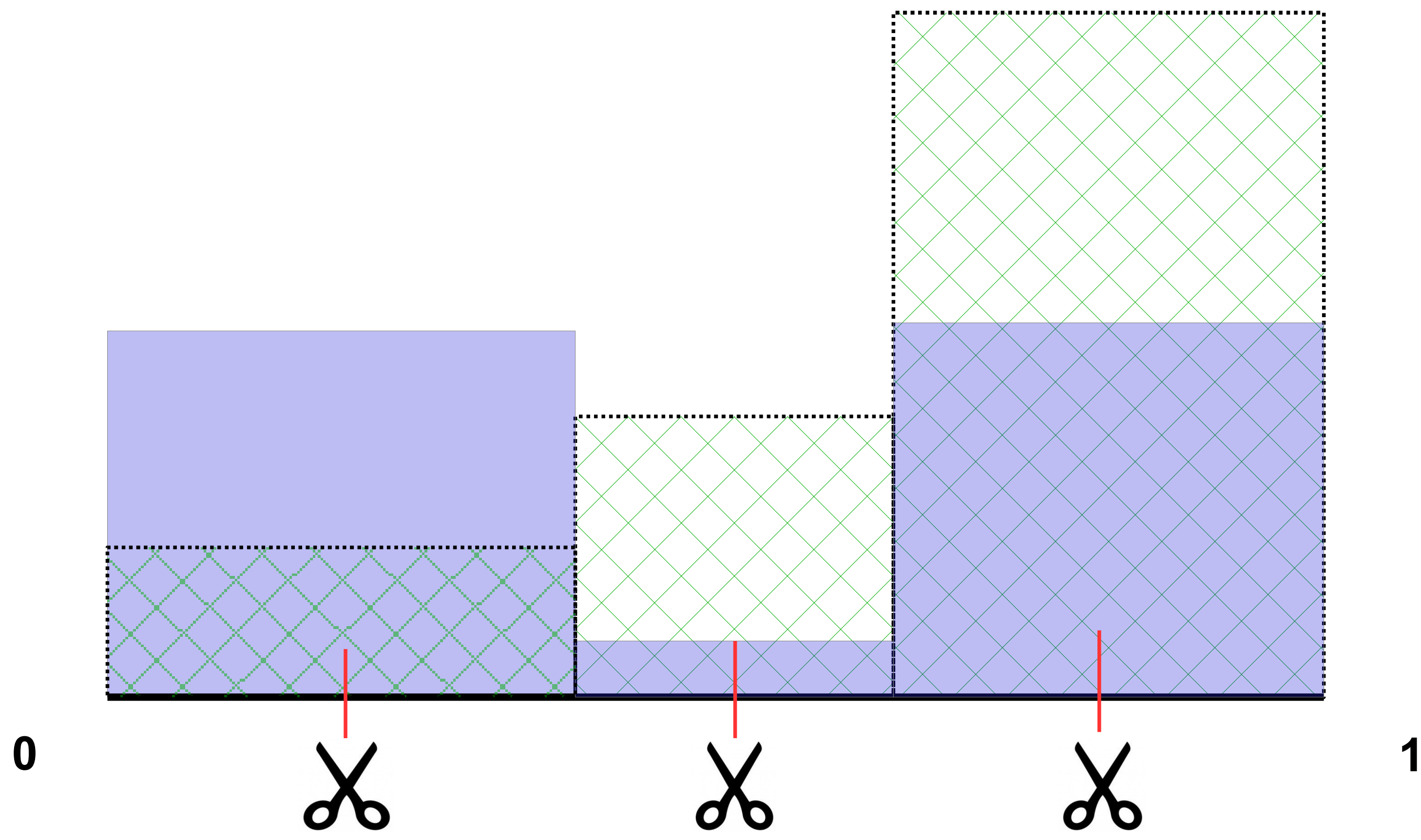
Envy-Freeness

- Allocation *A* is envy-free if no agent likes someone else's piece more than their own: $V_i(A_i) \geq V_i(A_j), \forall i,j \in N$

- Trivial envy-free allocation: throw away the entire cake
- Envy-freeness implies proportionality if the entire cake is allocated

Example

Two agents with piecewise constant valuations:

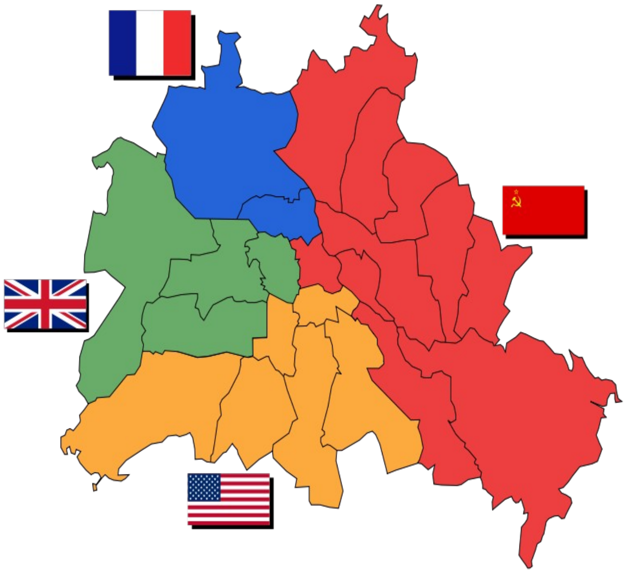


Additional Criteria

Bounded number of cuts

- Some protocols can allocate countable unions of crumbles

- An envy-free allocation with n-1 cuts is guaranteed to exist (but no finite protocol can find it)



Berlin divided by the
Potsdam Conference (1945)

Query Model (Robertson & Webb)

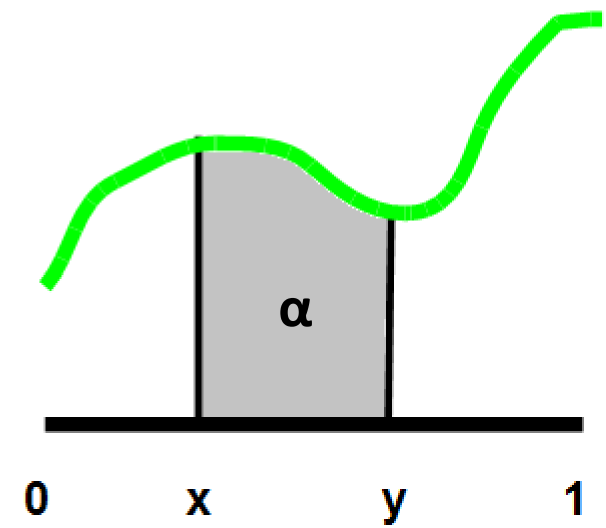
The agents are treated as oracles that can answer two types of queries:

$Cut_i(x, \alpha)$:

- Agent i returns $y \geq x$ such that $V_i([x, y]) = \alpha$

$Evaluate_i(x, y)$:

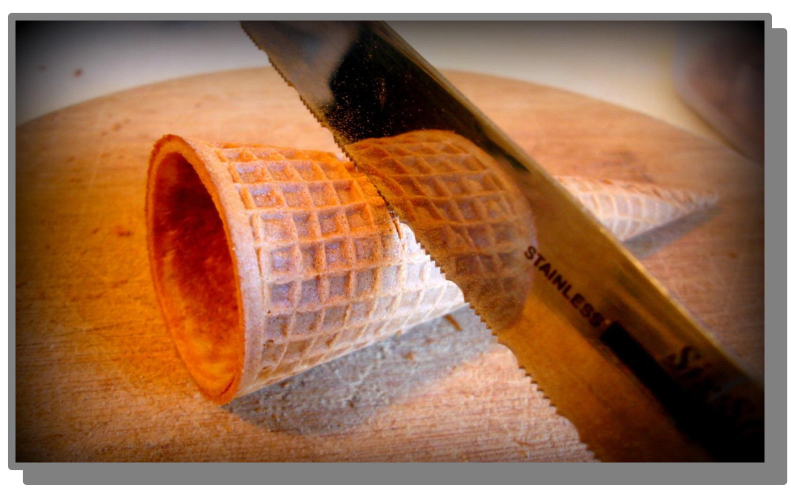
- Agent i returns α such that $V_i([x, y]) = \alpha$



Cut-and-Choose

Agent *1* cuts the cake in two equal pieces
Agent *2* chooses his favorite piece
Agent *1* takes the remainder

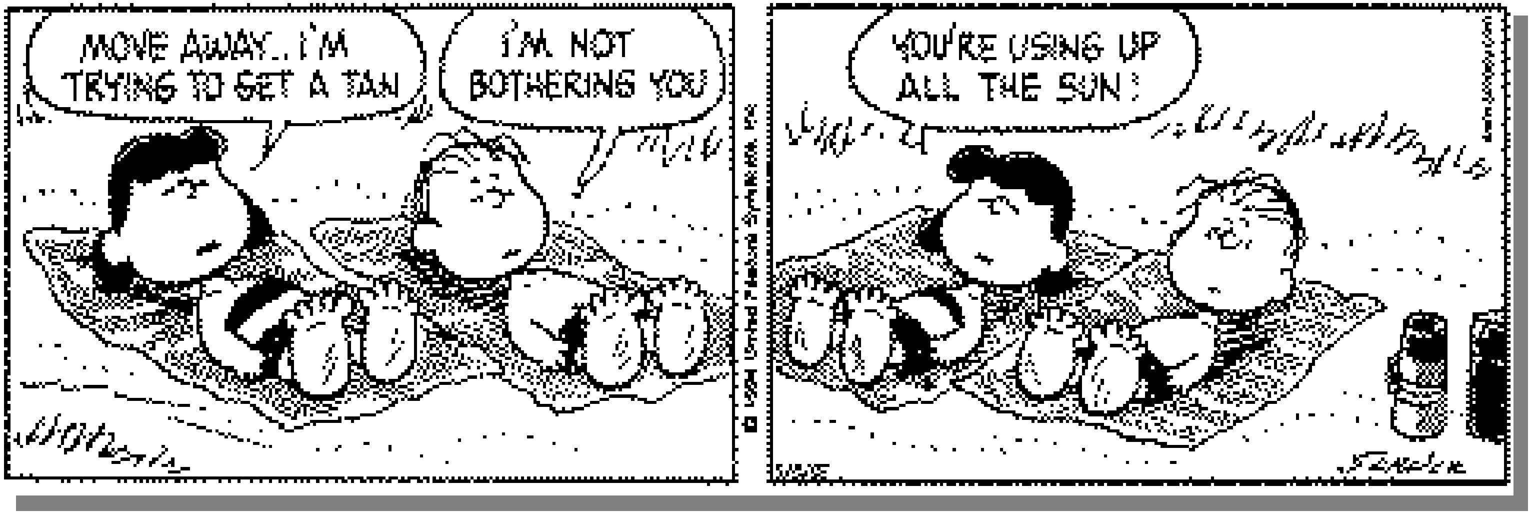
The allocation is proportional and envy-free



Externalities

- Also known as ***transaction spillovers:*** third parties are influenced by transactions they did not agree to

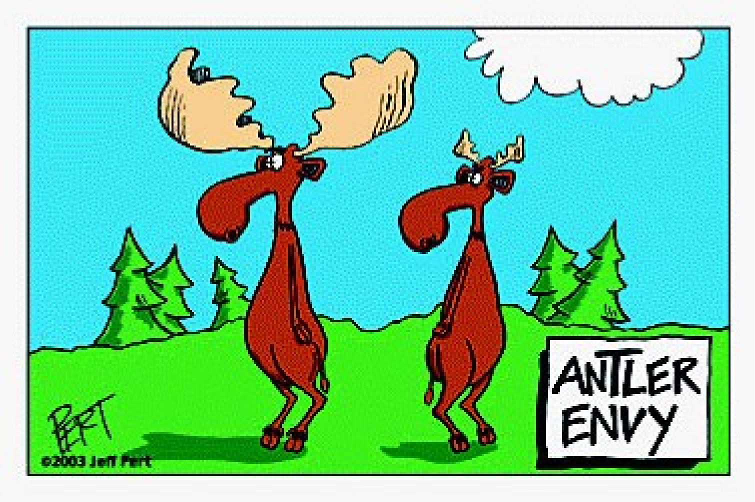
  - ***Negative externalities:*** pollution, smoking, risky choices (e.g. drinking and driving), overfishing
  - ***Positive externalities:*** education, immunization, environmental cleanup, research (!)



Externalities

- Common assumption in resource allocation: an agent's welfare is not affected by the consumption bundles of others (no externalities)

- ***Envy is related to externalities:*** agents naturally compare their own allocations with those of others.



This work:

- Generalized model to capture externalities due to synergies between agents (e.g. altruistic agents)

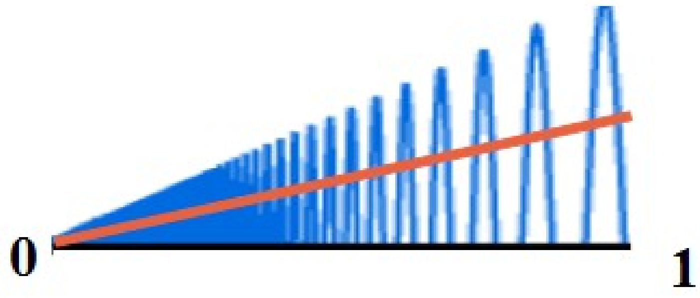
Externalities in Cake Cutting

Generalization:

- Each agent has several value density functions, such that $V_{i,j}(X)$ is the value agent i gets from the allocation of piece X to agent j

- The value of agent i under an allocation A is: $V_i(A) = \sum_{j=1}^{n} V_{i,j}(A_j)$

- The optimal allocation for one agent can require infinitely many cuts:



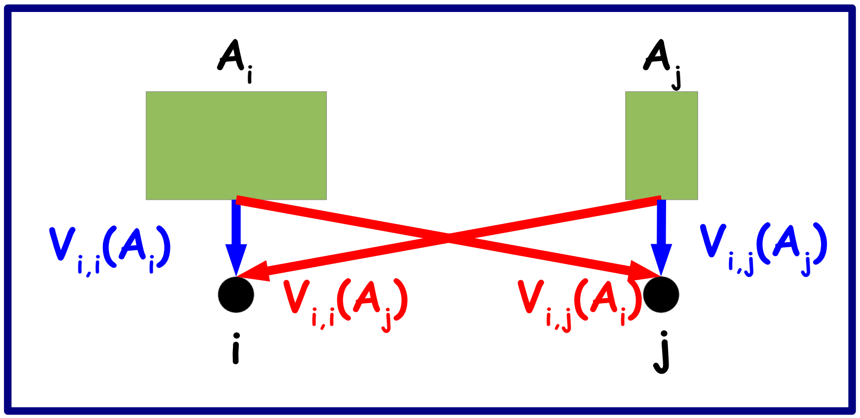
Fairness with Externalities

Proportionality: Allocation A is proportional if $V_i(A) \geq 1/n, \forall i \in N$

(the optimal allocation of an agent gives a value of 1)

Swap Envy-Freeness: Allocation A is swap envy-free if:

$$V_{i,i}(A_i) + V_{i,j}(A_j) \geq V_{i,i}(A_j) + V_{i,j}(A_i), \forall i,j \in N$$

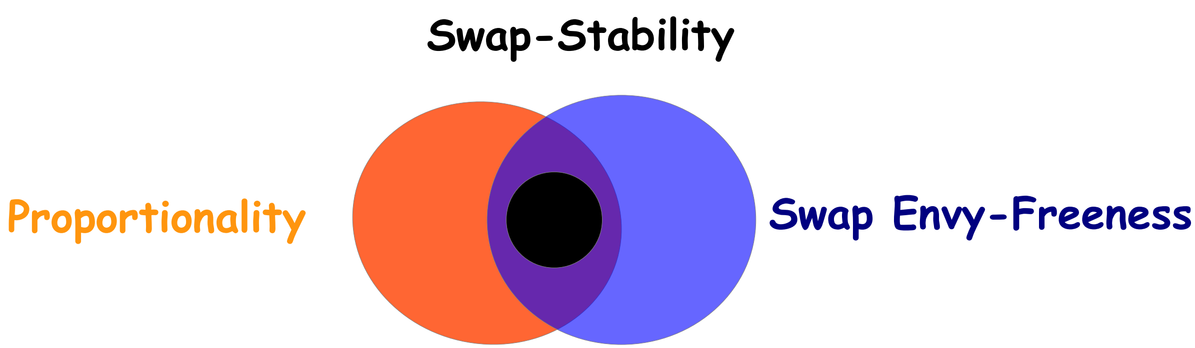


Swap Stability: Allocation A is swap-stable if:

$$V_{i,j}(A_j) + V_{i,k}(A_k) \geq V_{i,j}(A_k) + V_{i,k}(A_j), \forall i,j,k \in N$$

Existence of Fair Allocations and Bounds

- Swap-stability implies swap envy-freeness and proportionality; proportionality and swap envy-freeness are unrelated



Lower bound:

A swap envy-free and proportional allocation can require more than n-1 cuts

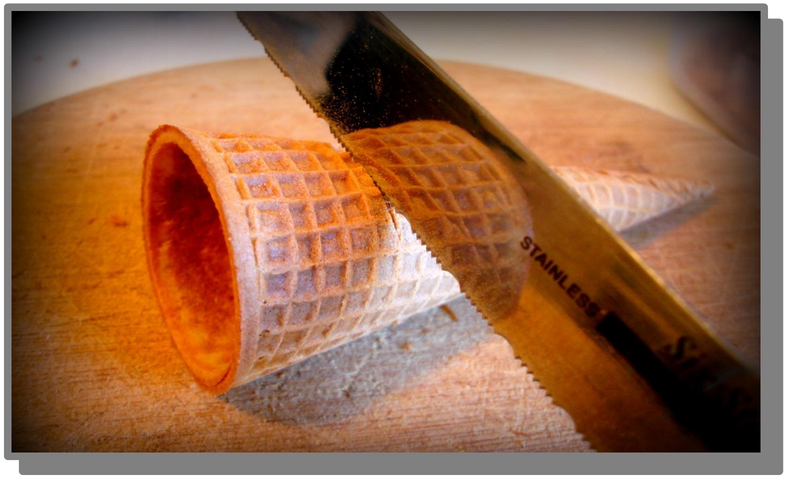
Upper bound:

Swap-stable allocations are guaranteed to exist and require at most $(n-1)n^2$ cuts when the value densities are continuous

Cut and Choose with Externalities

Agent 1 cuts at x such that
$V_{1,1}([0,x]) + V_{1,2}([x,1]) = V_{1,1}([x,1]) + V_{1,2}([0,x])$
Agent 2 chooses his favorite piece
Agent 1 takes the remainder

The allocation is proportional, swap envy-free, swap-stable



Query Model with Externalities

Extended Robertson-Webb:

- ***$Cut_{i,j}(x, \alpha)$:*** Agent i returns y such that $V_{i,j}([x, y]) = \alpha$
- ***$Evaluate_{i,j}(x, y)$:*** Agent i returns α such that $V_{i,j}([x, y]) = \alpha$

- **Theorem:** An allocation that guarantees $1/n^2$ to each agent can be computed with $O(n^2)$ queries in the extended Robertson-Webb model

Query Model with Externalities

Dubins-Spanier

A referee slides a knife across the cake, from left to right

When the knife reaches a point where one of the agents values the left piece at 1/n, that agent shouts ***CUT!***

The first agent to call cut receives the left piece and exits

Repeat with the remaining n-1 agents on the leftover cake (except now call cut at 1/(n-1) of the remainder)

Query Model with Externalities

Achieving $1/n^2$:

- Run Dubins-Spanier, but instruct each agent i to call stop when some agent j holds $1/n^2$ of its valuation: $V_{i,j}(X_{left}) = 1/n^2$
  - Allocate X_{left} (the piece to the left of the knife) to j and remove agent i from the game

- But there is no finite protocol that can compute a **<u>proportional</u>** allocation even for two agents in the extended Robertson-Webb model.

- What is the right query model then?

Query Model with Externalities

- ***Alternative Query Model:***
  - ***Cut Optimal$_i$(x, α):*** Agent i outputs y such that i's optimal allocation on [x, y], A_α, gives the agent exactly α: $V_i(A_\alpha) = \alpha$
  - ***Evaluate Optimal$_i$(x, y):*** Agent i outputs pair (A_α, α) such that A_α is an optimal allocation for i on [x, y] and $V_i(A_\alpha) = \alpha$

Theorem:

Every proportional protocol from the standard model translates to a proportional protocol with externalities when the *Cut* and *Evaluate* queries are replaced by *Cut Optimal* and *Evaluate Optimal*.

Discussion

- Tight bounds on the number of cuts (proportionality, swap envy-freeness, swap-stability)
- Query model and efficient protocols for swap-envy-freeness and swap-stability
- Negative externalities