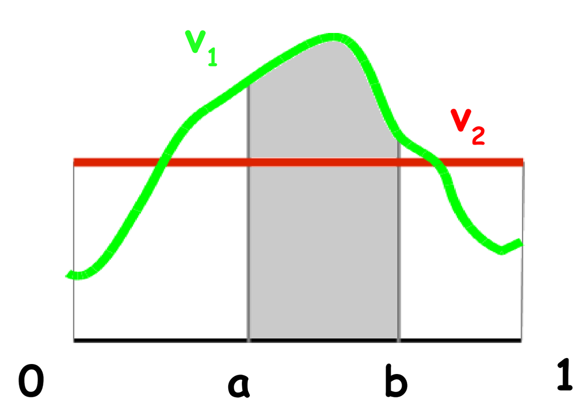
Externalities in Cake Cutting

Simina Brânzei — simina@cs.au.dk
Ariel Procaccia — arielpro@cs.cmu.edu
Jie Zhang — jiezhang@cs.au.dk

Cake Cutting

Fundamental problem in fair division; models the allocation of a divisible resource (time, land, computer memory) among agents with heterogeneous preferences.

- The cake is the interval [0, 1]
- Set of agents N = {1, ..., n}
- Each agent i has valuation function V_i over the cake, which is the integral of a value density function v_i



- A piece of cake is a finite union of disjoint subintervals of [0,1].
- The valuation of agent i for a piece X is given by the integral of their density function over the piece:

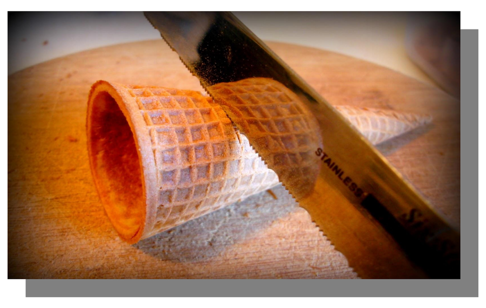
$$V_i(X) = \sum_{I \in X} \int_I v_i(x)\,dx$$

- An allocation A = (A_1, ..., A_n) is an assignment of pieces to agents such that each agent i receives piece A_i and all the A_i are disjoint.
- Allocation A is ***proportional*** if $V_i(A_i) \geq 1/n$, $\forall i \in N$ and ***envy-free*** if $V_i(A_i) \geq V_i(A_j)$, $\forall i,j \in N$.

Query Model (Robertson & Webb)

- All the discrete cake cutting protocols interact with the players using two types of queries:
  - ***$Cut_i(x, \alpha)$:*** Agent i returns y such that $V_i([x, y]) = \alpha$
  - ***$Evaluate_i(x, y)$:*** Agent i returns α such that $V_i([x, y]) = \alpha$

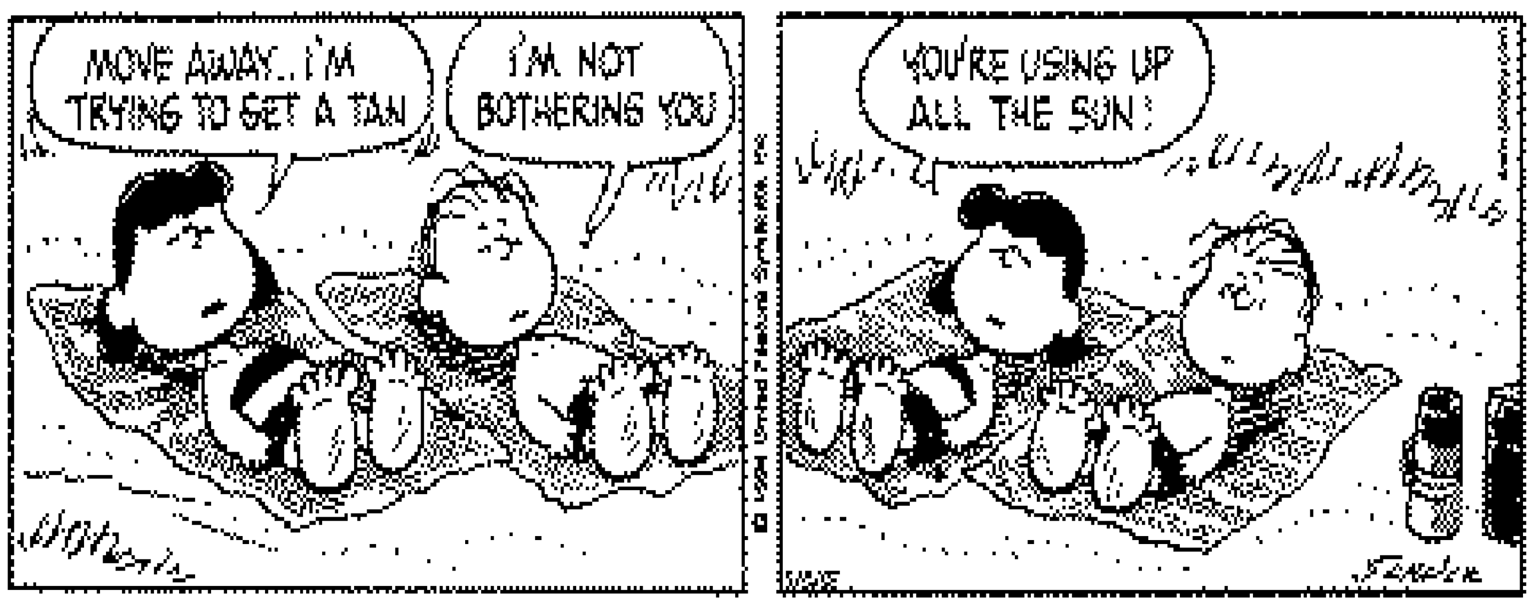
Example: Cut and Choose



Player 1 cuts the cake in two equal pieces
Player 2 chooses his favorite piece
Player 1 takes the remainder

Externalities

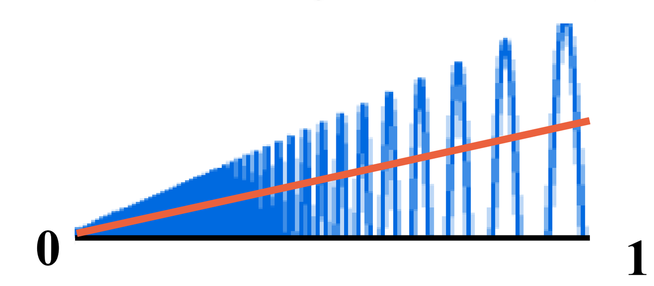
- Also known as ***transaction spillovers:*** third parties are influenced by transactions they did not agree to
  - ***Negative externalities:*** pollution, smoking, risky choices (e.g. drinking and driving), overfishing
  - ***Positive externalities:*** education, immunization, environmental cleanup, research (!)



- Common assumption in resource allocation: an agent's welfare is not affected by the consumption bundles of others (no externalities)
- *Envy is about externalities:* agents compare their own allocations with those of others.

Externalities in Cake Cutting

- Generalized model to capture externalities when there are synergies between agents (e.g. altruism).
- Each agent has several value densities: $V_{i,j}(X)$ is the value agent i gets from the allocation of piece X to agent j
- The value of agent i under an allocation A is: $V_i(A) = \sum_{j=1}^{n} V_{i,j}(A_j)$
- The optimal allocation of one agent can require infinitely many cuts:



Fairness Criteria

Proportionality:

- Allocation A is proportional if $V_i(A) \geq 1/n$, $\forall i \in N$

Swap Envy-Freeness:

- Allocation A is swap envy-free if

$$V_{i,i}(A_i) + V_{i,j}(A_j) \geq V_{i,i}(A_j) + V_{i,j}(A_i),\ \forall i,j \in N$$

Swap Stability:

- Allocation A is swap stable if

$$V_{i,j}(A_j) + V_{i,k}(A_k) \geq V_{i,j}(A_k) + V_{i,k}(A_j),\ \forall i,j,k \in N$$

Properties

Existence: Swap-stability implies swap envy-freeness and proportionality; swap envy-freeness and proportionality are unrelated

Lower bounds: A swap envy-free and proportional allocation can require strictly more than n-1 cuts

Upper bounds: Swap-stable allocations (which are also proportional and swap envy-free) are guaranteed to exist and require at most $(n-1)n^2$ cuts when the value densities are continuous

Query Model with Externalities

- Extended Robertson-Webb:
  - ***$Cut_{i,j}(x, \alpha)$:*** Agent i returns y such that $V_{i,j}([x, y]) = \alpha$
  - ***$Evaluate_{i,j}(x, y)$:*** Return α such that $V_{i,j}([x, y]) = \alpha$
- An allocation that guarantees $1/n^2$ to each agent can be computed with $O(n^2)$ queries in the extended Robertson-Webb model
- But there is no finite protocol that can compute a proportional allocation even for two agents in the extended Robertson-Webb model.
- Alternative:
  - ***Cut $Optimal_i(x, \alpha)$:*** Agent i outputs y such that i's optimal allocation on [x, y], A_α, gives the agent exactly α: $V_i(A_\alpha) = \alpha$
  - ***Evaluate $Optimal_{i,j}(x, y)$:*** Agent i outputs pair (A_α, α) such that A_α is an optimal allocation for i on [x, y] and $V_i(A_\alpha) = \alpha$
- Every proportional protocol from the standard model translates to a proportional protocol with externalities when the Cut and Evaluate queries are replaced by Cut Optimal and Evaluate Optimal.