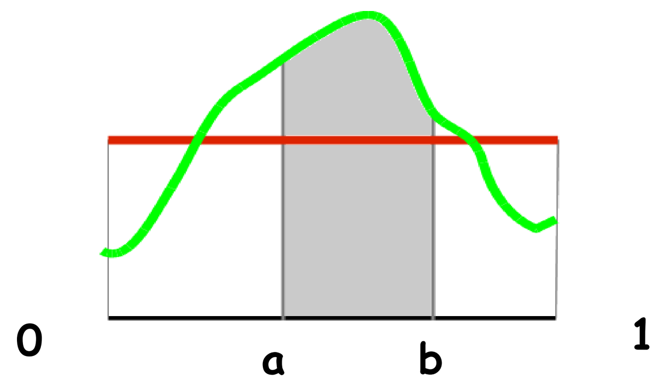
Equilibrium Analysis in Cake Cutting

Simina Brânzei and **Peter Bro Miltersen**
Aarhus University, Denmark
{simina, bromille}@cs.au.dk

Cake Cutting

Fundamental model in fair division; represents the problem of allocating a divisible resource (time, land, computer memory) among agents with heterogeneous preferences.

- The cake is the interval [0, 1]
- Set of agents N = {1, ..., n}
- Each agent i has a valuation function V_i over the cake, which is the integral of a value density function v_i



- A piece of cake X is a finite union of disjoint subintervals of [0,1]. A contiguous piece is a single subinterval
- The valuation of agent i for a piece X is given by the integral of their density function over the piece:

$$V_i(X) = \sum_{I \in X} \int_I v_i(x) dx$$

- An allocation X = (X_1, ..., X_n) is an assignment of pieces to agents such that each agent i receives piece X_i and all the X_i are disjoint.

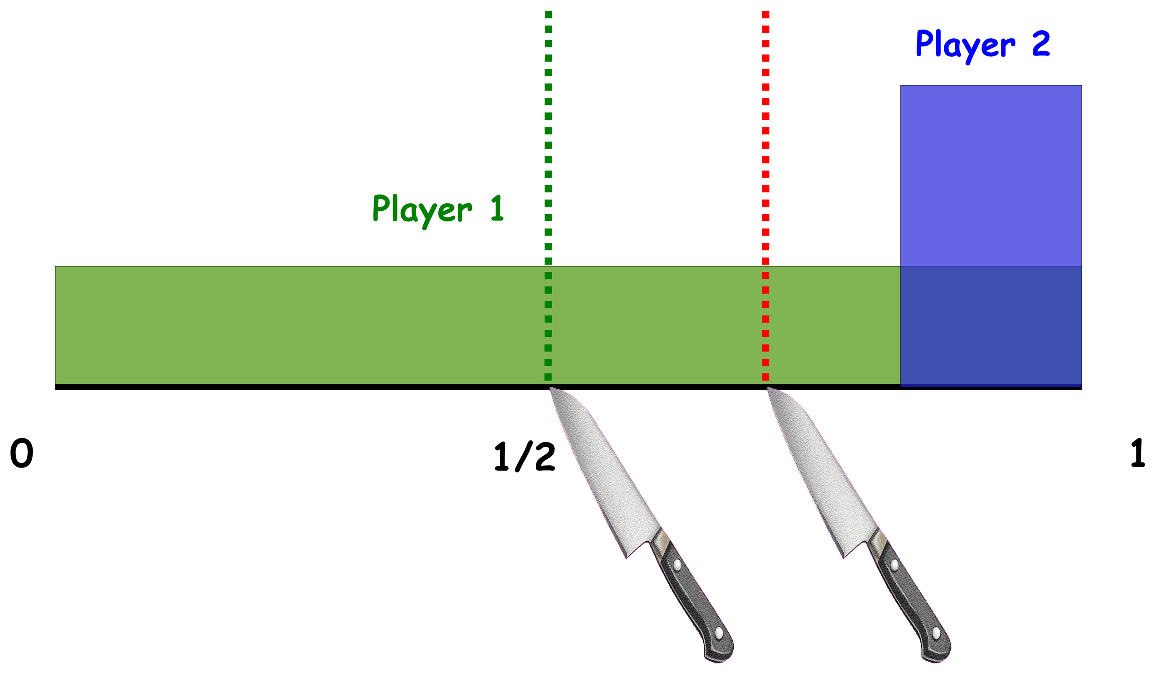
Fairness Criteria

Proportionality: An allocation X = (X_1, ... X_n) is proportional if $V_i(X_i) \geq 1/n$, $\forall$ i $\in$ N.
Envy-Freeness: An allocation X = (X_1, ... X_n) is envy-free if $V_i(X_i) \geq V_i(X_j)$, $\forall$ i,j $\in$ N.

Dubins-Spanier

- A referee slides a knife across the cake, from left to right
- When the knife reaches a point such that an agent values the piece to the left of the knife at 1/n, that agent shouts *CUT!*
- The first agent to call cut receives the left piece and exits
- Repeat with the remaining *n-1* agents on the leftover cake (except now call cut at *1/(n-1)* of the remainder)

- Dubins-Spanier guarantees proportionality (but not envy-freeness)
- The protocol is not strategyproof



Equilibrium Analysis

- While classical protocols are not necessarily strategyproof, they are very simple, natural, and can be implemented by the agents themselves by following a sequence of natural operations
- What do the equilibria of such protocols look like?

Moving Knife Game

- Knife moves across the cake, from left to right; the first agent to call cut receives the left piece and exits.
- Threshold strategies:
  - The strategy of each agent *i* is a vector
    $T_i = (t_{i,1}, ..., t_{i,n}) \in [0, 1]^n$
  - The agent calls cut in round *j* when the left piece is worth exactly t_{ij}.
- If multiple agents call cut simultaneously, break ties using a fixed permutation of N
- Complete information and strictly positive value density functions

Characterization of Nash Equilibria

Theorem: Consider a moving knife game with deterministic tie-breaking. Then every pure Nash equilibrium of the game induces an envy-free allocation that contains the entire cake.

- If an agent is envious of an earlier piece – just call cut faster in that round.
- If envious of a later piece, "skip" rounds until reaching that piece (by setting all intermediate thresholds as late as possible)

Theorem: Given any envy-free allocation with n-1 cuts, there exists a deterministic tie-breaking rule π such that the game has a pure Nash equilibrium inducing this allocation.

Theorem (characterization): A strategy profile *T* is a Nash equilibrium under a deterministic tie-breaking rule

if and only if

i). The induced allocation is envy-free,
ii). it contains the entire cake, and
iii). in every round except the last, the agent that is allocated the piece has an active competitor that calls cut simultaneously.

Theorem: For every $\varepsilon > 0$, the game has an ε-equilibrium that is independent of tie-breaking, which induces an ε-envy-free allocation that contains the entire cake.